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THE MOTION OF A PARTICLE IN THE NON-STATIONARY FIELD OF A LOGARITHMIC POTENTIAL†

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The plane motion of a material point, driven by a force inversely proportional to the distance from a fixed centre of variable mass, is studied. Consideration is given to the case in which the motion may be integrated by using a specially obtained first integral.

1. THE EQUATION of motion of a particle of unit mass in the field of a logarithmic potential is

$$z^{\cdot\cdot} + f(t)/\bar{z} = 0 \tag{1.1}$$

where $z(t) = x(t) + iy(t) = r(t)e^{i\varphi(t)}$ is the complex coordinate of the particle in the plane, t is the time $x(t)$, $y(t)$ are the Cartesian coordinates of the particle, $r(t)$, $\varphi(t)$ are its polar coordinates and $f(t)$ is a real function of time; the bar denotes complex conjugation. If $f(t)>0$ the particle is attracted to a centre of force at the origin; if $f(t)<0$ it is repelled. the total energy

$$E(t) = (x^{\cdot 2} + y^{\cdot 2})/2 + f(t)\ln\sqrt{x^2+y^2}$$

is not conserved, but the angular momentum

$$M = i(z\bar{z}^{\cdot} - \bar{z}z^{\cdot})/2 = r^2\varphi^{\cdot} \tag{1.2}$$

is an integral of the motion.

We shall study the case in which $p(t) = 1/f(t)$ is a quadratic polynomial in time:

$$p(t) = at^2 + bt + c$$

where a, b, c are real constants. Applying the method of [1] to Eq. (1.1), we can determine an integral of motion which is functionally independent of (1.2):

$$I = pz^{\cdot}\bar{z}^{\cdot} - p^{\cdot}(\bar{z}z^{\cdot} + z\bar{z}^{\cdot})/2 + p^{\cdot\cdot}z\bar{z}/2 + \ln(z\bar{z}/|p|) \tag{1.3}$$

After changing to polar coordinates this integral becomes

$$I = p(r^{\cdot 2} + M^2/r^2) - p^{\cdot}rr^{\cdot} + p^{\cdot\cdot}r^2/2 + \ln(r^2/|p|)$$

If the unknown function $r(t)$ is replaced by $u(t) = r^2(t)/p(t)$, we obtain the equation

$$p^2u^{\cdot 2} = -4u\ln|u| + 4Iu + Du^2 - 4M^2$$

where $D = b^2 - 4ac$ is the discriminant of $p(t)$.

Let us express the constant I as $I = \ln j$, $j>0$, and set $v = u/j$, $q = j^{1/2}p/2$, $\mu = M/j^{1/2}$, $\delta = jD/4$, $\alpha = aj^{1/2}/2$, $\beta = bj^{1/2}/2$, $\gamma = cj^{1/2}/2$. We will introduce a modified time variable

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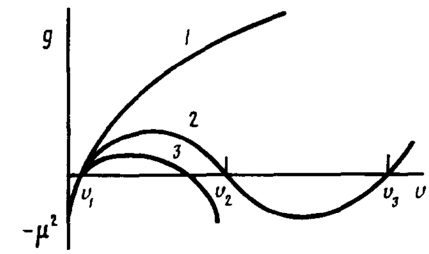

FIG. 1.

$$\tau = \int_{t_0}^{t} \frac{d\xi}{q(\xi)} = \begin{cases} \dfrac{1}{\sqrt{\delta}} \left| \ln \dfrac{(s-\sqrt{\delta})(s_0+\sqrt{\delta})}{(s+\sqrt{\delta})(s_0-\sqrt{\delta})} \right|, & \alpha \neq 0, \ \delta > 0 \\ \dfrac{2}{\sqrt{-\delta}} \left[\operatorname{arctg} \dfrac{s}{\sqrt{-\delta}} - \operatorname{arctg} \dfrac{s_0}{\sqrt{-\delta}} \right], & \delta < 0 \\ 2/s_0 - 2/s, & \alpha \neq 0, \ \delta = 0 \\ \beta^{-1} \ln | (\beta t + \gamma)/(\beta t_0 + \gamma) |, & \alpha = 0, \ \beta \neq 0 \\ (t - t_0)/\gamma, & \alpha = 0, \ \beta = 0 \end{cases} \tag{1.4}$$

$$(s = 2\alpha t + \beta, \ s_0 = 2\alpha t_0 + \beta)$$

where t_0 is the starting time. Since $d\tau/dt = 2/[p\sqrt{(j)}]$, the integrals of motion (1.2) and (1.3) may be written, respectively, as follows:

$$d\varphi/d\tau = {}^1/_2\mu/v \tag{1.5}$$

$$(dv/d\tau)^2 = -v \ln | v | + \delta v^2 - \mu^2 \tag{1.6}$$

2. The form of the solution of Eq. (1.6) depends on the shape of the graph of the function $g(v) = -v\ln|v| + \delta v^2 - \mu^2$ (Fig. 1), the initial value $v_0 = v(0)$ and the choice of sign for the initial velocity $dv(0)/d\tau$. We shall only discuss the case in which $t \geqslant t_0$ for all $p(t) > 0$. In that case we are only interested in those points of the domain $\geqslant 0$ at which $g(v \geqslant 0)$. If $\delta \geqslant 1/2$, then $g(v)$ has only one root v_1 (Fig. 1, curve 1). The coordinate $v(\tau)$ of the particle tends to $+\infty$ as τ increases. If the initial velocity is negative, then before going to $+\infty$ the point stops and turns at $v = v_1$. If $0 < \delta < 1/2$, then $g(v)$ has a local maximum and a local minimum (Fig. 1, curve 2) and, depending on the magnitude of μ^2, it will vanish once (in which case the motion will be similar to that considered previously), twice or three times. If there are two roots, one of them (w) is a double root: $g(w) = g'(w) = 0$, and therefore is a stationary point of the solution of Eq. (1.6) relative to the modified time τ. The corresponding solution of our original equation (1.1) is a spiral of the form:

$$r^2(t) = jp(t)\,w, \quad \varphi(t) = \varphi_0 + {}^1/_2\mu/w\,\tau(t), \quad t \geqslant t_0 \tag{2.1}$$

where φ_0 is the initial angle. If $w = v_1 = v_2 < v_3$, the solution (2.1) is stable; if $v_2 = v_3 = w > v_1$, it is unstable, and as $v(\tau) \to w$ the particle will execute a critical motion on the modified time scale. If there are three distinct roots $v_1 < v_2 < v_3$ (Fig. 1, curve 2), the particle may execute infinite motion in the domain $v \geqslant v_3$ and periodic motion in the interval between the roots v_1 and v_2. In that case the two spirals defined by (2.1) with $w = v_1$ and $w = v_2$, respectively, may be considered as two boundaries between which the true path of the particle will lie.

If $\delta \leqslant 0$, then $g(v)$, depending on μ^2, will have either no roots, one root or two (Fig. 1, curve 3). In the modified time scale, the particle may execute only periodic motion in the interval between the two roots. If the two roots v_1 and v_2 coincide, it will travel a stable spiral path. In particular, if $\delta = 0$, $\mu^2 = 1/e$, we obtain the path described by (2.1) with $w = 1/e$, where e is the base of the natural logarithm.

As is obvious from (1.4), an infinite interval $[t_0, \infty)$ on the ordinary time axis t may define a bounded interval $[0, \tau_1)$ on the modified time axis τ. For example if $\delta < 0$, $\alpha > 0$, we get

$$\tau_1 = \frac{1}{\sqrt{-\delta}}\left(\pi - 2\operatorname{arctg}\frac{s_0}{\sqrt{-\delta}}\right) \tag{2.2}$$

The solution of the equation

$$\tau_1 = \int_{v_0}^{w} \frac{d\xi}{\sqrt{-\xi \ln|\xi| + \delta\xi^2 - \mu^2}}$$

for w, substituted into (2.1), defines a spiral curve, to which the actual path of the particle will tend with time. In cases where the time τ_1 is finite, one should find an analytical solution of the non-linear equation as a Maclaurin series, using the technique of [2]. For Eq. (1.6) this gives an expansion

$$v(\tau) = v_0 + s_1 + s_2 + \ldots$$

The expressions for the first seven terms are as follows:

$$s_1 = A\tau, \quad A = (-v_0 \ln|v_0| + \delta v_0^2 - \mu^2)^{1/2}$$
$$s_2 = h_0\tau^2/2!, \quad h_0 = (2\delta v_0 - 1 - \ln|v_0|)/2$$
$$s_3 = Ah_1\tau^3/3!, \quad h_1 = (2\delta - 1/v_0)/2$$
$$s_4 = (h_0h_1 + A^2h_2)\,\tau^4/4!, \quad h_2 = 1/2v_0^2$$
$$s_5 = (A^3h_3 + A(h_1^2 + h_0h_2))\,\tau^5/5!, \quad h_3 = -1/v_0^3$$
$$s_6 = [A^4h_4 + A^2(6h_0h_3 + 5h_1h_2) + 3h_0^2h_2 + h_0h_1^2]\,\tau^6/6!, \quad h_4 = 3/v_0^4$$
$$s_7 = [A^5h_5 + A^3(h_0h_4 + 11h_1h_3 + 5h_2^2) + A(15h_0(h_0h_3 + h_1h_2) + h_1^2h_2 + h_1^3)]\,\tau^7/7!, \quad h_5 = -12/v_0^5$$

Figure 2 shows the results of solving equation (1.6) with $\delta = -11.700$; $\mu = 0.064$; $v_0 = 0.018$; $v_1 = 0.016$; $v_2 = 0.132$. Curve 1 was computed using the formula $v(\tau) = v_0 + s_1 + s_2 + \ldots + s_7$ and curve 2 by applying the algorithm of [3] with mesh-size $H = 0.001$. The special feature of this high-accuracy algorithm is that it enables one, while computing, to conserve the value of the first integral (1.3) of the equation of motion. In our example, the results are practically identical up to a time $\tau < 0.3$. Since $\delta = -11.7 < 0$, it follows from (1.4) and (2.2) that as real time varies from $t_0 = -\infty$ to $+\infty$ the modified time varies by an amount $\tau_1 = 2\pi/\sqrt{|\delta|} \simeq 1.8$. Hence our analytical formulas for the approximate solution may turn out to be applicable over a large real time interval.

Note that in the special case when $D = b^2 - 4ac = 0$ we obtain a result established by I. V. Meshcherskii.

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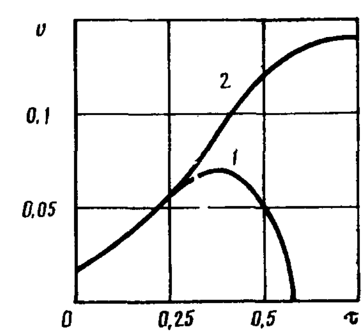

FIG. 2.

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ON THE EXISTENCE OF STATIONARY SOLITARY WAVES IN A ROTATING FLUID†

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A mathematical proof that there are no stationary solutions of the soliton type is given for a number of equations related to Ostrovskii's equation which, in particular, describes the surface and internal waves in a rotating fluid. A physical interpretation of this fact is presented. It is shown that, in the case of a different character of the high frequency dispersion which corresponds, for example, to capillary waves on a shallow rotating fluid, the conditions of the theorem are not satisfied as a result of which the prohibition on the existence of solitons is lifted. In this case, both single solitons as well as stationary formations consisting of solitons, that is, multisolitons, are constructed using numerical calculations.

1. FORMULATION OF THE PROBLEM

Consider the class of non-linear wave equations of the form

$$\frac{\partial}{\partial x}\left(\frac{\partial\eta}{\partial t} + c\frac{\partial\eta}{\partial x} + \frac{\alpha}{p}\frac{\partial\eta^p}{\partial x} + \beta\frac{\partial^3\eta}{\partial x^3}\right) = \gamma\eta \tag{1.1}$$

Here, $\eta(x, t)$ is an unknown function, c, α, β, γ and p are constants and $p > 1$. Equations belonging to this family are generated on the one hand by the generalised Korteweg–de Vries (KdV) equations and pass into them when $\gamma = 0$ and, on the other hand, their structure is close to the structure of the Kadomtsev–Petviashvili

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